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Covering and Piercing Disks with Two Centers

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Abstract

We give exact and approximation algorithms for two-center problems when the input is a set $\mathcal{D}$ of disks in the plane. We first study the problem of finding two smallest congruent disks such that each disk in $\mathcal{D}$ intersects one of these two disks. Then we study the problem of covering the set $\mathcal{D}$ by two smallest congruent disks.

1 Introduction

The standard *two-center problem* is a well known and extensively studied problem: Given a set P of n points in the plane, find two smallest congruent disks that cover all points in P. The best known deterministic algorithm runs in $O(n \log^2 n \log^2 \log n)$ [4] and there is a randomized algorithm with expected running time $O(n \log^2 n)$ [8]. There has also been a fair amount of work on several variations of the two-center problem: for instance, the two-center problem for weighted points [7], and for a convex polygon [17].

In this paper we consider new versions of the problem where the input consists of a set $\mathcal{D}$ of n disks (instead of points): In the *intersection problem* we want to compute two smallest congruent disks C_1 and C_2 such that each disk in $\mathcal{D}$ intersects C_1 or C_2, while in the *covering problem*, all disks in $\mathcal{D}$ have to be contained in the union of C_1 and C_2. To the best of our knowledge these problems have not been considered so far. However, linear-time algorithms are known for both the covering and the intersection problem with only one disk [9, 12, 13].

Our results. In order to solve the intersection problem, we first consider the two-piercing problem: Given a set of disks, decide whether there exist two points such that each disk contains at least one of these points. We show that this problem can be solved in $O(n^2 \log^2 n)$ expected time and $O(n^2 \log^2 \log \log n)$ deterministic time. Using these algorithms we can solve the intersection problem in $O(n^2 \log^3 n)$ expected time and $O(n^2 \log^4 n \log \log n)$ deterministic time.

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For the covering problem we consider two cases: In the *restricted case* each $D \in \mathcal{D}$ has to be fully covered by one of the disks C_1 or C_2. In the *general case* a disk $D \in \mathcal{D}$ can be covered by the union of C_1 and C_2. We show how the algorithms for the intersection problem can be used to solve the restricted covering case and present an exact algorithm for the general case. We complement these results by giving efficient approximation algorithms for both cases.

All the results presented in this paper are summarized in the following table.

	Exact algorithm	$(1+\epsilon)$-approximation
Intersection problem	$O(n^2 \log^4 n \log\log n)$ $O(n^2 \log^3 n)$ expected time	–
General covering problem	$O(n^3 \log^4 n)$	$O(n + 1/\epsilon^3)$
Restricted covering problem	$O(n^2 \log^4 n \log\log n)$ $O(n^2 \log^3 n)$ expected time	$O(n + (1/\epsilon^3) \log 1/\epsilon)$

Notation. The radius of a disk D is denoted by $r(D)$ and its center by $c(D)$. The circle that forms the boundary of D is denoted by ∂D.

Without loss of generality, we assume that no disk in $\mathcal{D}$ contains another disk in $\mathcal{D}$.

2 Intersecting Disks with Two Centers

In this section we consider the following intersection problem: Given a set of disks $\mathcal{D} = \{D_1, \ldots, D_n\}$, we want to find two smallest congruent disks C_1 and C_2 such that each disk $D \in \mathcal{D}$ has a nonempty intersection with C_1 or C_2.

Based on the observation below, there is an $O(n^3)$ algorithm for this problem.

Observation 1. *Let (C_1, C_2) be a pair of optimal covering disks. Let ℓ be the bisector of the segment connecting the centers of C_1 and C_2. Then, $C_i \cap D \neq \emptyset$ for every $D \in \mathcal{D}$ whose center lies on the same side of ℓ as the center of C_i, for $i = \{1, 2\}$.*

A simple approach would be, for every bipartition of the centers of the disks in $\mathcal{D}$ by a line ℓ, to compute the smallest disk intersecting the disks on each side of ℓ, and return the best result over all bipartitions. Since there are $O(n^2)$ such partitions, and the smallest disk intersecting a set of disks can be found in linear time [12], this algorithm runs in $O(n^3)$ time.

We will present faster algorithms for the intersection problem. We first introduce a related problem. For a real number $\delta \geq 0$ and a disk D, the δ-*inflated* disk $D(\delta)$ is a disk concentric to D and whose radius is $r(D) + \delta$. Consider the following decision problem:

> Given a value $\delta \geq 0$, are there two points p_1 and p_2 such that $D(\delta) \cap \{p_1, p_2\} \neq \emptyset$ for every $D \in \mathcal{D}$?

This problem is related to our original problem in the following way. The above condition holds with δ if and only if the two disks centered at p_1 and p_2 with radius δ intersect all disks $D \in \mathcal{D}$. Therefore the two disks centered at p_1 and p_2 with radius δ^* are a solution to the intersection problem, where δ^* is the minimum value for which the answer to the decision problem is "yes".

2.1 Decision Algorithm

Given a value $\delta \geq 0$, we construct the arrangement of the δ-inflated disks $D_i(\delta), i = 1 \ldots n$ in the plane. This arrangement consists of $O(n^2)$ cells, each cell being a $0, 1$, or 2-face. We traverse all the cells in the arrangement in a depth-first manner and do the followings: We place one center point, say p_1, in a cell. The algorithm returns "yes" if all the disks that do not contain p_1 have a nonempty common intersection. Otherwise, we move p_1 to a neighboring cell, and repeat the test until we visit every cell. This naïve approach leads to a running time $O(n^3)$: we traverse $O(n^2)$ cells, and each cell can be handled in linear time.

The following approach allows us to improve this running time by almost a linear factor. We consider a traversal of the arrangement of the δ-inflated disks by a path γ that crosses only $O(n^2)$ cells, that is, some cells may be crossed several times, but on average each cell is crossed $O(1)$ times. It can be achieved by choosing the path γ to be the Eulerian tour of the depth-first search tree from the naïve approach.

While we move the center p_1 along γ and traverse the arrangement, we want to know whether the set of disks $\mathcal{D}'$ that do not contain p_1 have a non-empty intersection. To do this efficiently, we use a segment tree [6]. Each disk of $\mathcal{D}$ may appear or disappear several times during the traversal of γ: each time we cross the boundary of a cell, one disk is inserted or deleted from $\mathcal{D}'$. So each disk appears in $\mathcal{D}'$ along one or several segments of γ. We store these segments in a segment tree. As there are only $O(n^2)$ crossings with cell boundaries along γ, this segment tree is built over a total of $O(n^2)$ endpoints and thus has total size $O(n^2 \log n)$: Each segment of γ along which a given disk of $\mathcal{D}$ is in $\mathcal{D}'$ is inserted in $O(\log n)$ nodes of the segment tree. Each node v of the segment tree stores a set $\mathcal{D}_v \subseteq \mathcal{D}$ of input disks; from the discussion above, they represent disks that do not contain p_1 during the whole segment of γ that is represented by v. In addition, we store at node v the intersection $I_v = \bigcap \mathcal{D}_v$ of the disks stored at v. Each such intersection I_v is a convex set bounded by $O(n)$ circular arcs, so we store them as an array of circular arcs sorted along the boundary of I_v. In total it takes $O(n^2 \log^2 n)$ time to compute the intersections I_v for all nodes v in the segment tree, since each disk is stored at $O(n \log n)$ nodes on average and the intersection of k disks can be computed in $O(k \log k)$ time.

We now need to decide whether at some point, when p_1 moves along γ, the intersection of the disks in $\mathcal{D}'$ (that is, disks that do not contain p_1) is nonempty. To do this, we consider each leaf of the segment tree separately. At each leaf, we test whether the intersection of the disks stored at this leaf and all its ancestors is non-empty. So it reduces to emptiness testing for a collection of $O(\log n)$ circular polygons with $O(n)$ circular arcs each. We can solve this in $O(\log^2 n)$ expected time by randomized convex programming [5, 16], using $O(\log n)$ of the following primitive operations:

1. Given I_i, I_j and vector $a \in \mathbb{R}^2$, find the extreme point $v \in I_i \cap I_j$ that minimizes $a \cdot v$.

2. Given I_i and a point p, decide whether $p \in I_i$.

We can also solve this problem in $O(\log^2 n \log \log n)$ time using deterministic convex programming [3]. So we obtain the following result:

Lemma 2. *Given a value $\delta \geq 0$, we can decide in $O(n^2 \log^2 n)$ expected time or in $O(n^2 \log^2 n \log \log n)$ worst-case time whether there exist two points such that every δ-inflated disk intersects at least one of them.*

2.2 Optimization Algorithm

The following lemma shows that the optimum δ^* can be found in a set of $O(n^3)$ possible values.

Lemma 3. *When $\delta = \delta^*$, p_1 or p_2 is a common boundary point of three δ^*-inflated disks, a tangent point of two δ^*-inflated disks or $\delta^* = 0$.*

Proof. Suppose that this is not the case. Then the common intersection of the disks containing p_1 has nonempty interior. Similarly, the common intersection of the disks containing p_2 has nonempty interior. Let p_1' and p_2' be points in the interiors, one from each common intersection. Then there is a value $\delta' < \delta$ satisfying $D(\delta') \cap \{p_1', p_2'\} \neq \emptyset$ for every $D \in \mathcal{D}$. But we also assumed that $\delta^* \neq 0$. □

Finding δ^*

Due to Lemma 3 we consider only discrete values of δ for which one of the events defined in Lemma 3 occurs. Whether $\delta^* = 0$ can be tested with the decision algorithm in $O(n^2 \log^2 n)$ expected time or in $O(n^2 \log^2 n \log\log n)$ worst-case time. So from now on, we assume that p_1 or p_2 is a common boundary point of three δ-inflated disks or a tangent point of two δ-inflated disks.

In order to compute all possible values for δ, we construct a frustum $f_i \in \mathbb{R}^3$ for each disk $D_i \in \mathcal{D}$. The bottom base of the frustum f_i is D_i lying in the plane $z = 0$. The intersection of f_i and the plane $z = \delta$ is $D_i(\delta)$. The top base of f_i is $D_i(\delta_{\max})$, where $\delta_{\max}$ is the minimum radius of the disk intersecting all disks in $\mathcal{D}$. Clearly, the optimal value of δ is in $[0, \delta_{\max}]$.

2.2.1 Event points and their corresponding radii.

Consider the case that $p_1 = (x, y)$ is the common boundary point of the disks $D_i(\delta)$, $D_j(\delta)$, and $D_k(\delta)$ in the plane. Then the point $p' = (x, y, \delta)$ is the common boundary point of three frustums f_i, f_j, and f_k. Consider now the case that $p_1 = (x, y)$ is the tangent point of $D_i(\delta)$ and $D_j(\delta)$. Then the point $p' = (x, y, \delta)$ is the point with the smallest z-value on the intersection curve of f_i and f_j. We call such a point the *tangent point* of two frustums. Hence, in order to find the points p_1 and p_2, all the tangent points and the common boundary points of the frustums have to be considered. There are $O(n^2)$ tangent points and $O(n^3)$ common boundary points, therefore there are $O(n^3)$ candidates for the point p_1 in total (note that for each candidate for p_1, the corresponding value for δ is obtained, namely the height of p_1'). Thus, a naïve way to find the minimum value δ such that there exists two points p_1, p_2 that fulfil the conditions, is to test all candidate δ values. For each possible δ value, we can determine if there are two points p_1, p_2 such that all $D(\delta)$ are intersected by p_2 or p_1 (as argued above). The solution is the smallest value δ^* at which the decision algorithm in Section 2.1 returns "yes". This leads to a running time of $O(n^5 \log^2 n)$ expected time or $O(n^5 \log^2 n \log\log n)$ deterministic time.

In order to improve the running time we use an implicit binary search.

2.2.2 Implicit binary search.

We perform an implicit binary search on the δ values corresponding to these common boundary points. As argued above, p_1 is the projection of a point p' which is a tangent point of two

frustums or a common boundary point of three frustums, i.e., a vertex of the arrangement $\mathcal{A}$ of the n frustums $f_1, \dots f_n$; the complexity of $\mathcal{A}$ is $O(n^3)$. We now describe how to perform the binary search over the vertices of $\mathcal{A}$ in an implicit way:

Binary search on a coarse list of events. We first consider $O(n^2)$ pairs of frustums and compute the tangent point of each pair. Then we randomly select $O(n^2 \log n)$ triples of frustums and compute the common boundary point of each triple. Since $\delta^* \in [0, \delta_{\max}]$, we only consider points whose z-value is in this interval. Clearly, these points are vertices of $\mathcal{A}$ and hence we randomly select $O(n^2 \log n)$ vertices from $\mathcal{A}$. We sort their radii associated with them in $O(n^2 \log^2 n)$ time. By a binary search with the decision algorithm in Section 2.1, we determine two consecutive radii δ_i and δ_{i+1} such that δ^* is between δ_i and δ_{i+1}. This takes $O(n^2 \log^3 n)$ time. Since the vertices were picked randomly, the strip $W[\delta_i, \delta_{i+1}]$ bounded by the two planes $z := \delta_i$ and $z := \delta_{i+1}$ contains only $k = O(n)$ vertices of $\mathcal{A}$ with high probability [14, Section 5].

Zooming into the interval. We compute all the k vertices in $W[\delta_i, \delta_{i+1}]$ by a standard sweep-plane algorithm in $O(k \log n + n^2 \log n)$ time as follows: First, we compute the intersection of the sweeping plane at $z := \delta_i$ with the frustums $f_1, \dots, f_n$. This intersection forms a two-dimensional arrangement of $O(n)$ circles with $O(n^2)$ total complexity, and we can compute it in $O(n^2 \log n)$ time. We next construct the portion of the arrangement $\mathcal{A}$ in $W[\delta_i, \delta_{i+1}]$ incrementally by sweeping a plane orthogonal to z-axis from the intersection at $z := \delta_i$ towards $z := \delta_{i+1}$. As a result, we can compute the $k = O(n)$ vertices (and the corresponding $O(n)$ radii) in $W[\delta_i, \delta_{i+1}]$ in $O(n \log n)$ time. We abort the sweep if the number k of vertices inside the strip becomes too large and restart the algorithm with a new random sample. This happens only with small probability. In order to find the minimum value δ^*, we perform a binary search on these $O(n)$ radii we just computed, using the decision algorithm in Lemma 2. This takes $O(n^2 \log^3 n)$ expected time. The solution pair of points p_1 and p_2 can also be found by the decision algorithm.

To get a deterministic algorithm, we use the parametric search technique, with the deterministic decision algorithm of Lemma 2. As the generic algorithm, we use an algorithm that computes in $O(\log n)$ time the arrangement of the inflated disks using $O(n^2)$ processors [2], so we need to run the decision algorithm $O(\log^2 n)$ times, and the total running time becomes $O(n^2 \log^4 n \log \log n)$.

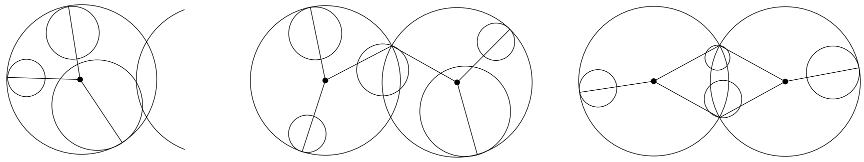
Theorem 4. *Given a set $\mathcal{D}$ of n disks in the plane, we can compute two smallest congruent disks whose union intersects every disk in $\mathcal{D}$ in $O(n^2 \log^3 n)$ expected time, and in $O(n^2 \log^4 n \log \log n)$ deterministic time.*

3 Covering Disks with Two Centers

In this section we consider the following covering problem: Given a set of disks $\mathcal{D} = \{D_1, \dots, D_n\}$, compute two smallest congruent disks C_1 and C_2 such that each disk $D \in \mathcal{D}$ is covered by C_1 or C_2. In the *general case*, a disk $D \in \mathcal{D}$ must be covered by $C_1 \cup C_2$. In the *restricted case*, each disk $D \in \mathcal{D}$ has to be fully covered by C_1 or by C_2.

3.1 The General Case

We first give a characterization of the optimal covering. The optimal covering of a set $\mathcal{D}'$ of disks by one disk is determined by at most three disks of $\mathcal{D}'$ touching the optimal covering disk such that the convex hull of the contact points contains the center of the covering disk. (See Figure 1(a).)



(a) (b) (c)

Figure 1: The three configurations for the optimal 2-center covering of disks.

When covering by two disks, a similar argument applies, and thus the optimal covering disks (C_1^*, C_2^*) are determined by at most five input disks.

Lemma 5. *The optimal covering by two disks C_1^*, C_2^* satisfies one of the following conditions.*

1. *For some $i \in \{1, 2\}$, the disk C_i^* is the optimal one-covering of the disks contained in C_i^*, as in Figure 1(a).*
2. *There is an input disk that is neither fully contained in C_1^* nor in C_2^*, but contains one point of $\partial C_1^* \cap \partial C_2^*$ in its boundary as in Figure 1(b).*
3. *There are two input disks D_i, D_j, possible $i = j$, none of them being fully covered by C_1^* or C_2^*, such that D_i contains one point of $\partial C_1^* \cap \partial C_2^*$ and D_j contains the other point of $\partial C_1^* \cap \partial C_2^*$ in their boundaries as in Figure 1(c).*

In all cases, each covering disk C^* is determined by at most three disks whose contact points contain the center $c(C^*)$ in their convex hull.

Proof. The optimal solution is a pair of congruent disks that achieves a local minimum in radius, that is, we cannot reduce the radius of the covering disks by translating them locally. If one covering disk is completely determined by the input disks contained in it, then this belongs to case 1. Otherwise, there always exists at least one input disk D such that D is not contained in C_i^* for all $i \in \{1, 2\}$. Moreover such input disks always touch $C_1^* \cup C_2^*$ from inside at the intersection points of ∂C_1^* and ∂C_2^*, otherwise we can always get a pair of smaller congruent covering disks. If only one point of $\partial C_1^* \cap \partial C_2^*$ is touched by an input disk D, both covering disks are determined by at most two additional disks touching from inside together with D because the covering disks are congruent. If both intersection points of $\partial C_1^* \cap \partial C_2^*$ are touched by input disks D_i and D_j, possible $i = j$, one covering disk is determined by one additional disk and the other covering disk by at most one additional disk touching from inside together with D_i and D_j because the covering disks are congruent. It is not difficult to see that there are two or three touching points of each covering disk that make radial angles at most π; otherwise we can get a pair of smaller congruent covering disks. $\square$

Using a decision algorithm and the parametric search technique, we can construct an exact algorithm for the general covering problem.

Let r^* be the radius of an optimal solution for the general case of covering by two disks. We describe a decision algorithm based on the following lemma that, for a given $r > 0$, returns "yes" if $r \geq r^*$, and "no" otherwise. (See also Figure 2).

Lemma 6. *Assume that $r \geq r^*$. Then there exists a pair of congruent disks C_1, C_2 of radius r such that their union contains the input disks, an input disk D touches C_1 from inside, and one of the following property holds.*

(a) C_1 is identical to D.

(b) There is another input disk touching C_1 from inside.

(c) There is another input disk D' such that D' is not contained in C_2, but it touches a common intersection t of ∂C_1 and ∂C_2 that is at distance $2r$ from the touching point of D. If this is the case, we say that D and t are aligned *with respect to C_1.*

(d) There are two disks D_i and D_j, possibly $i = j$, such that D_i touches a common intersection of ∂C_1 and ∂C_2, and D_j touches the other common intersection of ∂C_1 and ∂C_2.

Proof. Let c_1^* and c_2^* be the centers of the optimal solution. Imagine that we place two disks at c_1^* and c_2^* with radius larger than r^*, respectively. If C_1 is already identical to an input disk D, it belongs to case (a). Otherwise we translate C_1 towards C_2 until it hits an input disk D. Then we rotate C_1 around D in clockwise orientation maintaining D touching C_1 from inside until the union of C_1 and C_2 stops covering the input. If this event is caused by another disk touching C_1 from inside, it belongs to case (b). Otherwise the event is caused by another disk D_i that is hit by one of two common intersections of ∂C_1 and ∂C_2 at t. If D and t are aligned with respected to C_1, it belongs to case (c).

Otherwise, we rotate C_2 around t in counterclockwise until the union of C_1 and C_2 stops covering the input. If this event is caused by another disk D_j that is hit by the common intersection of ∂C_1 and ∂C_2, other than t, then it belongs to case (d). Otherwise the event is caused by another disk D' touching C_2 from inside. Thus, D touches C_1 from inside, D' touches C_2 from inside, and D_i touches the common intersection t of ∂C_1 and ∂C_2. Imagine that we rotate C_1 slightly further around D in clockwise. We also rotate C_2 around D' simultaneously such that D_i and the rotated copies keep maintaining a common intersection along their boundaries during the rotation. Let t denote the common intersection. We rotate C_1 and C_2 in such a way until we encounter an event (1) that another disk D_j touches C_1 or C_2, (2) that D_j touches the other common intersection of ∂C_1 and ∂C_2, or (3) that D and t are aligned with respect to C_1 or D' and t are aligned with respect to C_2. Note that if the event is of type (3), then D_i is not contained in the disk centered at $c(D)$ with radius $2r - r(D)$ or is not contained in the disk centered at $c(D')$ with radius $2r - r(D')$ as in Figure 2. □

3.1.1 Decision Algorithm.

The cases are enumerated as in Lemma 6.

Case (a). Choose an input disk D. C_1 has radius r and covers only D. Then C_2 is the smallest disk containing $\mathcal{D} \setminus D$. If the radius of C_2 is $\leq r$, we return "yes".

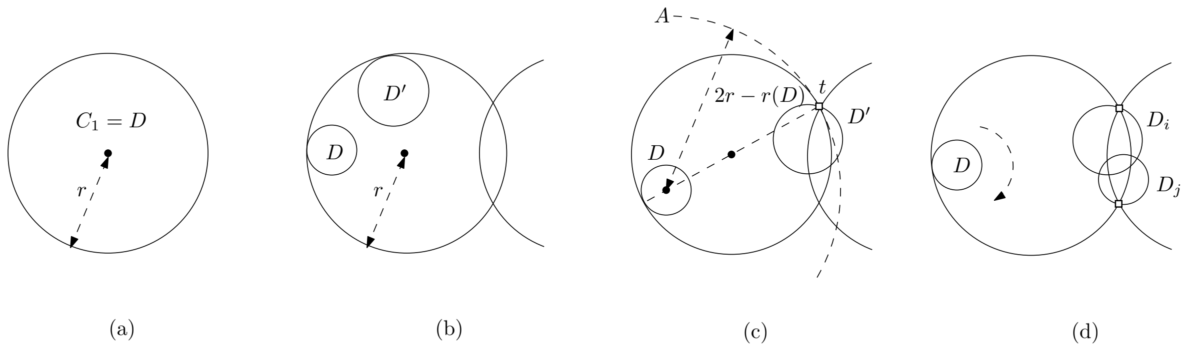


Figure 2: Four cases for $r \geq r^*$.

Case (b). We simply choose a pair of input disks D and D'. There are two candidates for C_1, as C_1 has radius r and touches D and D'. So we consider separately each of the two candidates for C_1. Then C_2 is chosen to be the smallest disk containing the input disks, or the portions of input disks (crescents) that are not covered by C_1, which can be done in $O(n)$ time. If for one of the two choices of C_1, the corresponding disk C_2 has radius $\leq r$, we return "yes".

Case (c). For each input disk D, we do the following.

1. For the circle A with center $c(D)$ and radius $2r - r(D)$, compute $A \cap D'$ for every other disk D'. Let t be such an intersection point.
2. For each t,
   (a) remove (part of) the input disks covered by the covering disk determined by D and t, and compute the smallest disk covering the remaining input.
   (b) If this algorithm returns a covering disk with radius $\leq r$, return "yes".

Case (d). For each input disk D that touches C_1 from inside, we do the following. Let i be the index of the first input disk that the circular arc of C_1 from the touching point hits in clockwise orientation. Let j be the index of the last input disk that the circular arc leaves. We claim that the number of pairs of type (i, j) is $O(n)$.

This claim can be easily proved by observing that, while we rotate C_1 around an input disk D in clockwise orientation, C_1 sweeps the plane and the input disks in such a manner that the first input disk intersected by the arc of C_1 from the tangent point in clockwise orientation changes only $O(n)$ times; To see this, consider the union of the input disks, which consists of $O(n)$ circular arcs. The last input disk intersected also changes $O(n)$ times. So the pairing along the rotation can be done by scanning two lists (first and last) of disks. For each pair (i, j), we still have some freedom of rotating C_1 around D within some interval (C_2 changes accordingly.) During the rotation, an input disk not covered by the union of C_1 and C_2 may become fully covered by the union, or vice versa. We call such an event an *I/O event*. Note that an I/O event occurs only when an input disk touches C_1 or C_2 from inside. Again, we claim that the number of I/O events for each pair (i, j) is $O(n)$.

For this claim, consider a pair (i, j). During the rotation, the first intersection point moves along the boundary of disk D_i and the last intersection point moves along the boundary of disk D_j. Therefore, the movement of C_2 is determined by these two intersection points.

Clearly C_1 has at most $2(n-1)$ I/O events. For C_2, the trajectory of its center is a function graph which "behaves well" – Since it is a function on the radii of disks D_i and D_j, and their center locations, it is not in a complicated form (and its degree is low enough) that there are only $O(n)$ events.

We compute all I/O events and sort them. At the beginning of the rotation of C_1 around D, we compute the number of input disks that are not fully covered, and set the variable *counter* to this number. Then we handle I/O events one by one and update the counter. If the counter becomes 0, we return "yes".

In total, case (d) can be handled in $O(n^3 \log n)$ time.

Lemma 7. *Given a value $r > 0$, we can decide in $O(n^3 \log n)$ time whether there exists two disks with radius r that cover a set of given disks in the plane.*

For the optimization algorithm we use parametric search.

To use the parametric search technique, we will design a parallel version of the decision algorithm. Then the overall algorithm runs in time $O(p \cdot T_p + T_p \cdot T_d \log p)$, where p denotes the number of processors, T_d denotes the running time of a decision algorithm, and T_p denotes the running time of the parallel decision algorithm using p processors. We have a parallel decision algorithm where $p = O(n^3)$ processors and $T_p = O(\log^2 n)$ time for some constant $c > 1$. Thus the overall algorithm runs in time $O(n^3 \log^4 n)$ time.

Parallel decision algorithm. For case (a), we assign $O(n)$ processors to each candidate D. The 1-center disk covering for the disks in $\mathcal{D} \setminus D$ can be computed by a known parallel linear programming algorithm [11] in $O(\log^2 n)$ time with $O(n)$ processors. Hence, we can solve case (a) in $O(\log^2 n)$ time with $O(n^2)$ processor.

For case (b), we assign $O(n)$ processors to each pair (D, D') of input disks. With a covering disk C_1 of radius r determined by D and D', we cover the input disks and compute the crescents of the input disks not covered by C_1 in a constant time. The 1-center disk covering for the crescents can be computed in $O(\log^2 n)$ time with $O(n)$ processors [11]. Thus we can handle the case (b) in $O(\log^2 n)$ time with $O(n^3)$ processors, and moreover we can deal with case (c) in a similar way.

For case (d), we first compute the union U of all input disks in $O(\log n)$ time with $O(n^3)$ processors [15]. Next we assign $O(n^2)$ processors to each input disk D. We fix D. As C_1 rotates around D while they keep touching as in Figure 2(c), we need to figure out $O(n)$ pairs (i, j) of input disks such that D_i and D_j are the first and the last ones intersected by C_1, respectively. Such D_i and D_j must be on the boundary of U, so it is sufficient to consider the disks whose arcs appear on the boundary of U. To get these pairs, we assign $O(n)$ processors to each disk on the union boundary in order to calculate two rotating angles of C_1 at which C_1 hits the disk at the first and the last in $O(1)$ time. We collect all these angles, sort them, and extract the pairs (i, j) from the sorted list; all steps are easily done in $O(\log n)$ time using $O(n)$ processors.

For a fixed angle interval I determined by some pair (i, j), the set of input disks not covered by C_1 remains same, and we can also know which disks are those ones. Using $O(n)$ processors in $O(1)$ time for each input disk D' not covered by C_1, we compute the subintervals $J \subseteq I$ such that $C_1 \cup C_2$ determined by J contains D' at any angle in J. These subintervals are defined by I/O events we mentioned in the sequential decision algorithm, so there are $O(n)$ subintervals. Finally we test whether the intersection of the subintervals is empty or

not. If it is not empty, then it means there is a rotation angle in I at which all input disks not covered by C_1 get to be contained in $C_1 \cup C_2$. Otherwise, no angles in I guarantee the full coverage by $C_1 \cup C_2$. This test can be done in bottom-up fashion in $O(\log n)$ time using $O(n)$ processors. After testing all pairs (i, j), if there is a pair such that the intersection is not empty, then return "yes". This is done in $O(\log n)$ time with $O(n^2)$ processors for a fixed disk d touching C_1 from inside. Summing up all things, we can solve case (d) in $O(\log n)$ time with $O(n^3)$ processors.

Theorem 8. *Given a set of n disks in the plane, we can find a pair of congruent disks with smallest radius whose union covers all of them in $O(n^3 \log^4 n)$ time.*

3.1.2 Constant Factor Approximation.

We apply the well known greedy k-center approximation algorithm by Gonzalez [10] to our general covering case. It works as follows: First pick an arbitrary point c_1 in the union $\bigcup \mathcal{D}$ of our input disks. For instance, we could choose c_1 to be the center of D_1. Then compute a point $c_2 \in \bigcup \mathcal{D}$ that is farthest from c_1. This can be done in linear time by brute force. These two points are the centers of our two covering disks, and we choose their radius to be as small as possible, that is, the radius of the two covering disks is the maximum distance from any point in $\bigcup \mathcal{D}$ to its closest point in $\{c_1, c_2\}$. This algorithm is a 2-approximation algorithm, so we obtain the following result:

Theorem 9. *We can compute in $O(n)$ time a 2-approximation for the general covering problem for a set $\mathcal{D}$ of n disks.*

3.1.3 $(1+\epsilon)$-Approximation.

Our $(1+\epsilon)$-approximation algorithm is an adaptation of an algorithm by Agarwal and Procopiuc [1]. We start by computing a 2-approximation for the general covering case in $O(n)$ time using our algorithm from Theorem 9. Let C_1, C_2 be the disks computed by this approximation algorithm and let r be their radius. We consider a grid of size $\delta = \lambda \epsilon r$ over the plane, where λ is a small enough constant. That is, we consider the points with coordinates $(i\delta, j\delta)$ for some integers i, j. Observe that there are only $O(1/\epsilon^2)$ grid points in $C_1 \cup C_2$. The center of each disk D is moved to a nearby grid point. That is, a center (x, y) is replaced by $(\delta\lceil x/\delta \rceil, \delta \lceil y/\delta \rceil)$. If two or more centers are moved to the the same grid point, we only keep the disk with the largest radius. All the centers are now grid points inside $C_1 \cup C_2$, or at distance at most $\sqrt{2}\delta$ from the boundary of this union, so we are left with a set of $O(1/\epsilon^2)$ disks. We now replace this new set of disks by grid points: each disk is replaced by the grid points which are closest to the boundary of this disk and lie inside this disk. In order to compute these points we consider each column of the grid separately: The intersection of each disk with this column is an interval, and we replace the interval by the lowest and the highest grid point lying inside this interval. Since the set of disks has size $O(1/\epsilon^2)$ and the number of columns is $O(1/\epsilon)$, it takes in total $O(1/\epsilon^3)$ time. The set of grid points we obtain is denoted by P_g and its size is $O(1/\epsilon^2)$. We compute two smallest disks E_1, E_2 that cover P_g in $O(\frac{1}{\epsilon^2} \log^2 \frac{1}{\epsilon} \log^2 \log \frac{1}{\epsilon})$ time using the algorithm from Chan [4]. Choosing the constant λ small enough and increasing the radii of E_1, E_2 by $2\sqrt{2}\delta$, these disks are a $(1+\epsilon)$-approximation of the solution to our general disk cover problem.

Theorem 10. *Given a set $\mathcal{D}$ of n disks in the plane, a $(1+\epsilon)$-approximation for $\mathcal{D}$ in the general covering case can be computed in $O(n + 1/\epsilon^3)$ time.*

3.2 The Restricted Case

Observation 1 can be adapted to the restricted covering case.

Observation 11. *Let ℓ be the bisector of an optimal solution C_1 and C_2. Then, $D \subset C_i$ for every $D \in \mathcal{D}$ whose center lies in the same side of ℓ as the center of C_i, for $i = \{1, 2\}$.*

Hence, the restricted covering problem can be solved in $O(n^3)$ time, since for a set of n disks $\mathcal{D}$ the smallest disk covering all $D \in \mathcal{D}$ can be computed in $O(n)$ time [13] and there are $O(n^2)$ different bipartitions of the centers of the disks.

The algorithm from Section 2 can also be adapted to solve the restricted covering problem. We consider the decision problem, which can be formulated as follows: Given a set of n disks $\mathcal{D}$ and a value δ, we want to decide whether there exists two disks C_1, C_2 with radius δ, such that each disk $D_i \in \mathcal{D}$ is covered by either C_1 or C_2. This implies that for each disk $D_j \in \mathcal{D}$ covered by C_i, the following holds: $d(c(D_j), c(C_i)) + r(D_j) \leq \delta$, for $i = \{1, 2\}$. Let $r_{\max}$ be the maximum of radii of all disks in $\mathcal{D}$. It holds that $\delta \geq r_{\max}$, since if $\delta < r_{\max}$ there clearly exists no two disks with radius δ which cover $\mathcal{D}$. We can formulated the problem in a different way.

> Given a value δ, do there exist two points, p_1 and p_2, in the plane such that $D^*(\delta) \cap \{p_1, p_2\} \neq \emptyset$ for every $D \in \mathcal{D}$, where $D^*(\delta)$ is a disk concentric to D and whose radius is $\delta - r(D) \geq 0$.

Recall the definition of δ-*inflated* disks from Section 2. Every disk $D \in \mathcal{D}$ was replaced by a disk concentric to D and whose radius was $r(D) + \delta$. Here we actually need to replace each disk D by a disk that is concentric to D and has a radius $\delta - r(D)$. Since we know that $\delta \geq r_{\max}$, we add an initialization step, in which every disk D is replaced by a disk concentric to D and whose radius is $r_{\max} - r(D)$. Then we can use exactly the same algorithm in Section 2 in order to compute a solution for the restricted covering problem. Let δ^* be the solution value computed by this algorithm. Clearly the solution for the covering problem is then $\delta^* + r_{\max}$. We summarize this result in the following theorem.

Theorem 12. *Given a set of n disks $\mathcal{D}$ in the plane, we can compute two smallest congruent disks such that each disk in $\mathcal{D}$ is covered by one of the disks in $O(n^2 \log^3 n)$ expected time or in $O(n^2 \log^4 n \log \log n)$ worst-case time.*

3.2.1 Constant Factor Approximation.

Let C_1, C_2 denote an optimal solution to the general case, and let r_g be their radius. Then any solution to the restricted case is also a solution to the general case, so we have r_g is at most the radius of the optimal solution to the restricted case. On the other hand, the inflated disks $C_1(2r_g), C_2(2r_g)$ form a solution to the restricted case, because any disk contained in $C_1 \cup C_2$ should be contained in either $C_1(2r_g)$ or $C_2(2r_g)$. So we obtain a 6-approximation algorithm for the restricted case by first applying our 2-approximation algorithm for the general case (Theorem 9) and then multiplying by 3 the radius of the two output disks:

Theorem 13. *Given a set of n disks $\mathcal{D}$ in the plane, we can compute in $O(n)$ time a 6-approximation to the restricted covering problem.*

As in the general case, we will see below how to improve it to a linear time algorithm for any constant approximation factor larger than 1.

3.2.2 $(1+\epsilon)$-Approximation.

Recall Observation 11. Let ℓ be the bisector of an optimal solution. Then each disk $D \in \mathcal{D}$ is covered by the disk C_i whose center lies in the same side of the center of C_i, $i \in \{1, 2\}$. Hence, if we know the bisector, we know the bipartition of the disks. First, we show how to compute an optimal solution in $O(n \log n)$ time if the direction of the bisector is known. Later on we explain how this algorithm is used in order to obtain a $(1+\epsilon)$ approximation.

Fixed Orientation. W.l.o.g, assume that the bisector is vertical. After sorting the centers of all $D \in \mathcal{D}$ by their x-values, we sweep a vertical line ℓ from left to right, and maintain two sets $\mathcal{D}_1$ and $\mathcal{D}_2$: $\mathcal{D}_1$ contains all disks whose centers lie to the left of ℓ and $\mathcal{D}_2 = \mathcal{D} \setminus \mathcal{D}_1$. Let C_1 be the smallest disk covering $\mathcal{D}_1$ and C_2 the smallest disk covering $\mathcal{D}_2$. While sweeping ℓ from left to right, the radius of C_1 is nondecreasing and the radius of C_2 nonincreasing and we want to compute $\min \max(r(C_1), r(C_2))$. Hence, we can perform a binary search on the list of the centers of the disks in $\mathcal{D}$. Each step takes $O(n)$ time, thus we achieve a total running time of $O(n \log n)$.

Sampling. We use $2\pi/\epsilon$ sample orientations chosen regularly over 2π, and compute for each orientation the solution in $O(n \log n)$ time. The approximation factor can be proven by showing that there is a sample orientation that makes angle at most ϵ with the optimal bisector. Without loss of generality, we assume that the bisector, denoted by b, of an optimal solution C_1^* and C_2^* is vertical as in Figure 3. Let q denote the midpoint of the segment connecting $c(q_1)$ and $c(q_2)$.

Let ℓ be the line which passes through q and makes angle with b is at most ϵ in counter-clockwise direction as in the figure. (For simplification we set the angle in the calculation to exactly ϵ.) Let p denote the intersection point of ℓ with the upper circular arc of ∂C_1^*, and let p' denote the point symmetric to p along b. Clearly p' lies on the boundary of C_2^*. We will show that there exists two disks C_1, C_2 where C_1 covers all disks whose centers lie to the left of ℓ and C_2 covers all disks whose centers lie to the right of ℓ and $r(C_1) = r(C_2) \leq (1+\epsilon)r(C_1^*)$.

We will explain the construction of C_2 and prove that C_2 covers all disks whose centers lie to the right of ℓ. C_1 can be constructed analogously. The center of C_2 is set to $c(C_2^*)$ and the radius is set to $|c(C_2^*)p| \leq |\overline{c(C_2^*)p'}| + |\overline{p'p}|$. It holds that $|\overline{c(C_2^*)p'}| + |\overline{p'p}| \leq r(C_2^*) + 4r(C_2^*) \sin\epsilon$, since $|\overline{p'q}| \leq 2r(C_2^*)$ and the distance of p' to b is at most $2r(C_2^*)\sin\epsilon$. Clearly C_2 covers all disks that were covered by C_2^*. In addition, it must cover all disks whose centers lie in the region of C_1^* that is bounded by ℓ and b and that has q as its lowest point, depicted as the dark gray region in Figure 3. Note that the disks whose centers lie in this region are fully covered by C_1^*, but not necessarily by C_2^*.

It remains to prove that all disks having their center in the dark gray region are fully covered by C_2. Let C' be the disk symmetric to C_1^* along ℓ. Then all disks whose centers lie in the dark gray region are covered by $C_1^* \cap C'$, because this region is symmetric along ℓ and

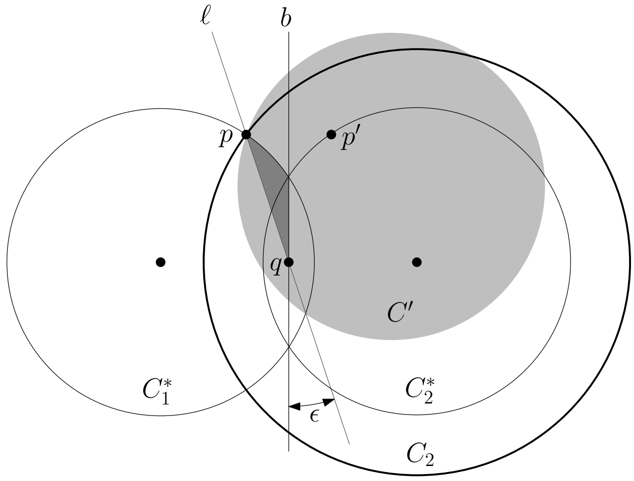


Figure 3: $r(C_2) \le (1+\epsilon')r(C_2^*)$ for any $\epsilon' \ge 4\epsilon$.

they are fully covered by C_1^*. Since C_2 contains the intersection $C_1^* \cap C'$, we conclude that all disks whose centers lie on the right side of ℓ are covered by C_2.

We can prove the analog for C_1. Hence,

$$r(C_1) = r(C_2) \le (1 + 4\sin\epsilon)r(C_1^*) \le (1+\epsilon')r(C_1^*) = (1+\epsilon')r(C_2^*)$$

as $\sin\epsilon \le \epsilon$ for $\epsilon \le 1$ (can be shown by using the theory of Taylor series) and for any $\epsilon' \ge 4\epsilon$. Since any solution whose bisector is parallel to ℓ has a radius at most $r(C_1)$, this solution has radius at most $(1+\epsilon')$ times the optimal radius.

Theorem 14. *For a given a set $\mathcal{D}$ of n disks in the plane, a $(1+\epsilon)$ approximation for the restricted covering problem for $\mathcal{D}$ can be computed in $O((n/\epsilon)\log n)$ time.*

The running time can be improved to $O(n + 1/\epsilon^3 \log 1/\epsilon)$ in the following way. We start with computing a 6-approximation in $O(n)$ time, using Theorem 13. Let C_1' and C_2' be the resulting disks, and let r' be their radius. As in the proof of Theorem 10, we round the centers of all input disks $D \in \mathcal{D}$ to grid points inside $C_1' \cup C_2'$, with a grid size $\delta' = \lambda'\epsilon r'$, for some small enough constant λ'. Then we apply our FPTAS from Theorem 14 to this set of rounded disks and inflate the resulting disks by a factor of $\sqrt{2}\delta$. These disks are a $(1+\epsilon)$-approximation for the optimal solution. As there are only $O(1/\epsilon^2)$ rounded disks, this can be done in $O((1/\epsilon^3)\log 1/\epsilon)$ time.

Theorem 15. *For a given a set $\mathcal{D}$ of n disks in the plane, a $(1+\epsilon)$ approximation for the restricted covering problem for $\mathcal{D}$ can be computed in $O(n + (1/\epsilon^3)\log 1/\epsilon)$ time.*

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